Kuwait University | Math 102 | December 16, 2004
Dept of Math & Comp Sci | Second Examination | Duration: 75 minutes

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The use of calculator is not allowed. Turn off your mobile phone or pager

(a) Which indeterminate form does the function $(3-2\tanh x)^{\sinh x}$ have as $x \to \infty$, and state why? **(1 points)**

(b) Find $\lim_{x\to 0^+}\left(\dfrac{3^{\pi x}}{\tan x}-\dfrac{\cos^2 x}{\sin x}\right)$, if it exists. **(2 points)**

Find the partial fractions decomposition of

$$f(x)=\frac{x^5+2x^4+x^3-4x^2+2}{x^4+2x^3+2x^2}$$

(3 points)

Evaluate the following integrals. **(3 points each)**

(a) $\displaystyle\int (\cosh x)\sin x\,dx$

(b) $\displaystyle\int \frac{x}{(6x-8-x^2)^{\frac{3}{2}}}\,dx$

(c) $\displaystyle\int \frac{\sec^4 x}{\sqrt[3]{\tan x}}\,dx$

(d) $\displaystyle\int \frac{\sinh x}{\sqrt{\sinh^2 x-3}}\,dx$

(e) $\displaystyle\int \frac{2+\tan\frac{x}{2}}{2\sin x+2\cos x+3}\,dx$

Does the integral $\displaystyle\int_1^{\infty}\frac{1}{x\sqrt{6x-1}}\,dx$ converge or diverge? If it converges, find its value.

(4 points)

1. (a) $(3-2\tanh x)^{\sinh x}$ has the indeterminate form 1^{∞} as $x \to \infty$ since

$\lim\limits_{x\to\infty}(3-2\tanh x) = 3 - 2\lim\limits_{x\to\infty}\tanh x = 3-2=1$, and $\lim\limits_{x\to\infty}\sinh x = \infty$.

(b) $\frac{3^{\pi x}}{\tan x} - \frac{\cos^2 x}{\sin x}$ has the form $\infty - \infty$ at 0.

$$\lim_{x\to 0^+}\left(\frac{3^{\pi x}}{\tan x} - \frac{\cos^2 x}{\sin x}\right) = \lim_{x\to 0^+}\left(\frac{3^{\pi x}}{\tan x} - \frac{\cos x}{\tan x}\right) \overset{\text{L'Hospital's}}{=} \lim_{x\to 0^+}\frac{(\pi\ln 3)3^{\pi x} - \sin x}{\sec^2 x} = \pi \ln 3$$

2. $\frac{x^5+2x^4+x^3-4x^2+2}{x^4+2x^3+2x^2} = x + \frac{2-x^3-4x^2}{x^4+2x^3+2x^2}$, and

$$\frac{2-x^3-4x^2}{x^4+2x^3+2x^2} = \frac{2-x^3-4x^2}{x^2(x^2+2x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+2} = \frac{Ax(x^2+2x+2)+B(x^2+2x+2)+(Cx+D)x^2}{x^2(x^2+2x+2)}$$

$$-x^3-4x^2+2 = (A+C)x^3 + (2A+B+D)x^2 + (2A+2B)x + 2B$$

$$A+C=-1 \quad 2A+B+D=-4 \quad 2A+2B=0 \quad 2B=2 \Rightarrow \boxed{B=1,\quad A=-1,\quad D=-3,\quad C=0}$$

$$\frac{x^5+2x^4+x^3-4x^2+2}{x^4+2x^3+2x^2} = x - \frac{1}{x} + \frac{1}{x^2} - \frac{3}{x^2+2x+2}$$

3. (a) $I = \int (\cosh x)\sin x\,dx, \qquad u=\cosh x \quad dv = \sin x, \qquad du = \sinh x\,dx \quad v = -\cos x$

$$\begin{aligned} I &= \int u\,dv = uv - \int v\,du = -\cosh x\cos x + \int \sinh x \cos x\,dx = -\cosh x\cos x + \int \sinh x\,(\sin x)'\,dx \\ &= -\cosh x \cos x + \sinh x \sin x - I \\ I &= -\tfrac{1}{2}\cosh x\cos x + \tfrac{1}{2}\sinh x \sin x + C \end{aligned}$$

(b) $$\int \frac{x}{(6x-8-x^2)^{\frac{3}{2}}}\,dx = \int \frac{x}{(1-(x-3)^2)^{\frac{3}{2}}}\,dx \overset{x-3=\sin\theta}{=} \int \frac{\sin\theta+3}{\cos^3\theta}\cos\theta\,d\theta = \int(\tan x\sec\theta + 3\sec^2\theta)\,d$$

$$= \sec\theta + 3\tan\theta + C = \frac{3x-8}{\sqrt{6x-8-x^2}} + C$$

(c) $$\int \frac{\sec^4 x}{\sqrt[3]{\tan x}}\,dx = \int (\tan x)^{-1/3}\sec^2 x \times \sec^2 x\,dx = \int (\tan x)^{-1/3}(1+\tan^2 x)\sec^2 x\,dx$$

$$\overset{u=\tan x}{=} \int u^{-1/3}(1+u^2)du = \frac{3}{2}u^{2/3} + \frac{3}{8}u^{8/3} + C = \frac{3}{2}\tan^{2/3}x + \frac{3}{8}\tan^{8/3}x + C$$

(d) $u = \tan\frac{x}{2}$, $dx = \frac{2du}{1+u^2}$, $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$

$$\int \frac{2+\tan\frac{x}{2}}{2\sin x + 2\cos x + 3}\,dx = \int \frac{2+u}{\left(\frac{4u}{1+u^2} + 2\frac{1-u^2}{1+u^2}+3\right)}\frac{2}{1+u^2}\,du = \int \frac{4+2u}{u^2+4u+5}\,du = \ln\left|u^2+4u+5\right| + C$$

$$= \ln\left|\tan^2\tfrac{x}{2} + 4\tan\tfrac{x}{2} + 5\right| + C$$

(e) $$\int \frac{\sinh x}{\sqrt{\sinh^2 x - 3}}\,dx = \int \frac{\sinh x}{\sqrt{\cosh^2 x - 4}}\,dx \overset{u=\cosh x}{=} \int \frac{du}{\sqrt{u^2-4}} \overset{u=2\sec\theta}{=} \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\cosh x}{2} + \frac{\sqrt{\cosh^2 x - 4}}{2}\right| + C$$

4. $$\int \frac{1}{x\sqrt{6x-1}}\,dx \overset{u^2=6x-1}{=} \int \frac{2u}{u(u^2+1)}\,du = 2\tan^{-1}\sqrt{6x-1}$$

$$\int_1^{\infty} \frac{1}{x\sqrt{6x-1}}\,dx = \lim_{c\to\infty}\int_1^c \frac{1}{x\sqrt{6x-1}}\,dx = 2\lim_{c\to\infty}\left(\tan^{-1}\sqrt{6c-1} - \tan^{-1}\sqrt{5}\right) = \pi - 2\tan^{-1}\sqrt{5}$$